# *Continued Fractions and the Mandelbrot Set*

The Mandelbrot Set has been called the most complex objects in all of mathematics and its hidden secrets have fascinated professionals and amateurs alike ever since its discovery by Robert Brooks and Peter Matelski in 1978. But even its simplest features reveal astonishing relationships the most surprising of which is the Farey Fraction rule. It turns out that each of the 'buds' or 'lobes' which sprout from the main cardioid can be assigned a rational fraction called the rotation number a selection of which can be seen in Fig 1:

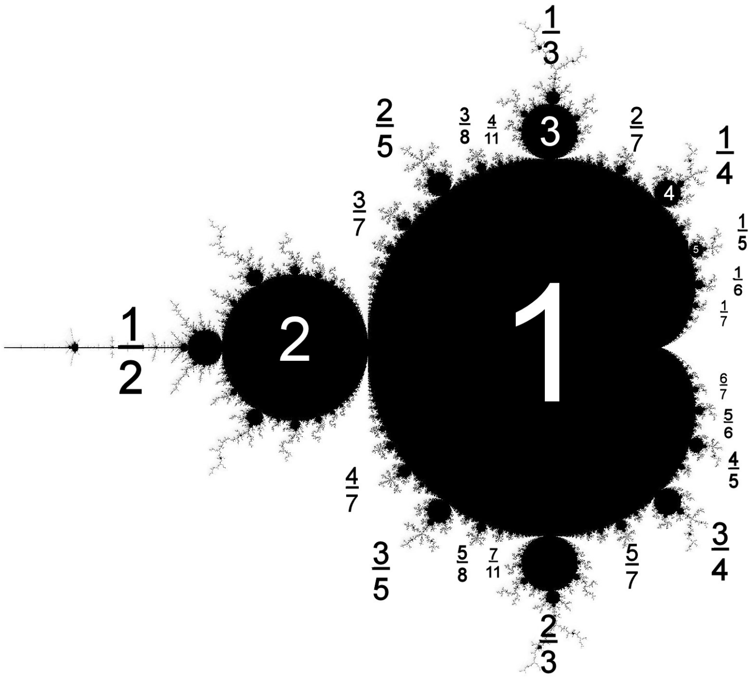


*Fig 1: The Mandelbrot set with a selection of rotation numbers*

The first thing to note is that the lobes fall naturally into sequences of diminishing size, the most obvious of which is the sequence of lobes which I have labelled 2, 3, 4, 5 … (Lobe number 1 is, of course, a special case.) For now you can think of the rotation number as a measure of 'how far round' the cardioid the lobe is. Lobe 2 is half way round. Lobe 3 looks as if it is only a quarter of the way round but this is because lobe 1 is a cardioid, not a circle.

The next thing to note is that the 'simpler' the fraction is the larger is the lobe to which it refers. Why is this?

And most mysterious of all. Why do the sequences obey Farey addition? Look at the sequence of diminishing lobes stretching from lobe 3 towards lobe 2. The rotation numbers are

$$1/2 \ \ldots\ldots \leftarrow 4/9 \leftarrow 3/7 \leftarrow 2/5 \leftarrow 1/3$$

To generate each member of the sequence you add 1 to the numerator and 2 to the denominator. Alternatively we can say that the rotation number of any member of a sequence is the Farey sum[1] of the previous member of the sequence and the rotation number of the lobe towards which the sequence is heading (1/2 in this case).

As an even more striking example, consider the lobes which start at 1/2 and stretch towards 2/3. The sequence is

$$1/2 \rightarrow 3/5 \rightarrow 5/8 \rightarrow 7/11 \rightarrow \ldots\ldots 2/3$$

In order to explain what is going on we need to examine the way the rational fraction between 0

1 The Farey sum or mediant of two fractions $a/c$ and $b/d$ is equal to $(a + b)/(c + d)$

and 1 are organised along the number line.

The 'simplest' fractions are obviously 1/2, 1/3, 1/4, 1/5. etc. Symmetry dictates that the complementary fractions (1/2,) 2/3, 3/4, 4/5 etc. are equally 'simple'

0 … 1/5 1/4 1/3 1/2 2/3 3/4 4/5 … 1

$$0 \quad ... \quad 0+\frac{1}{5} \quad 0+\frac{1}{4} \quad 0+\frac{1}{3} \quad 0+\frac{1}{2} \quad 1-\frac{1}{3} \quad 1-\frac{1}{4} \quad 1-\frac{1}{5} \quad ... \quad 1$$

We can usefully call these '*first order*' fractions. Note that the middle fraction can be written in two ways – either 0 + 1/2 or 1 – 1/2.

We can now proceed to define the '*second order*' fractions in the following way. Consider the gap between 1/2 and 1/3. The 'simplest' fraction which lies between these two is obviously 1/(2 + 1/2) – 2½ being half way between 2 and 3 . This is 2/5. This is, in fact the start of a whole sequence of fractions

$$\frac{1}{3} \quad ... \quad 0+\frac{1}{3-\frac{1}{5}} \quad 0+\frac{1}{3-\frac{1}{4}} \quad 0+\frac{1}{3-\frac{1}{3}} \quad 0+\frac{1}{2+\frac{1}{2}} \quad 0+\frac{1}{2+\frac{1}{3}} \quad 0+\frac{1}{2+\frac{1}{4}} \quad 0+\frac{1}{2+\frac{1}{5}} \quad ... \quad \frac{1}{2}$$

which is the sequence

1/3 … 5/14 4/11 3/8 2/5 3/7 4/9 5/11 … 1 /2

Now without going into too much detail, suffice it to say that what we are doing is expressing every rational fraction as a '*nearest integer continued fraction*'. (In a standard continued fraction we always use the integer part and a positive remainder. In a NICF we use the nearest integer and accept that some of the 'remainders' will be negative.) We can, in fact, build up a complete 'tree' of rational fractions in this way as shown in Fig. 2.

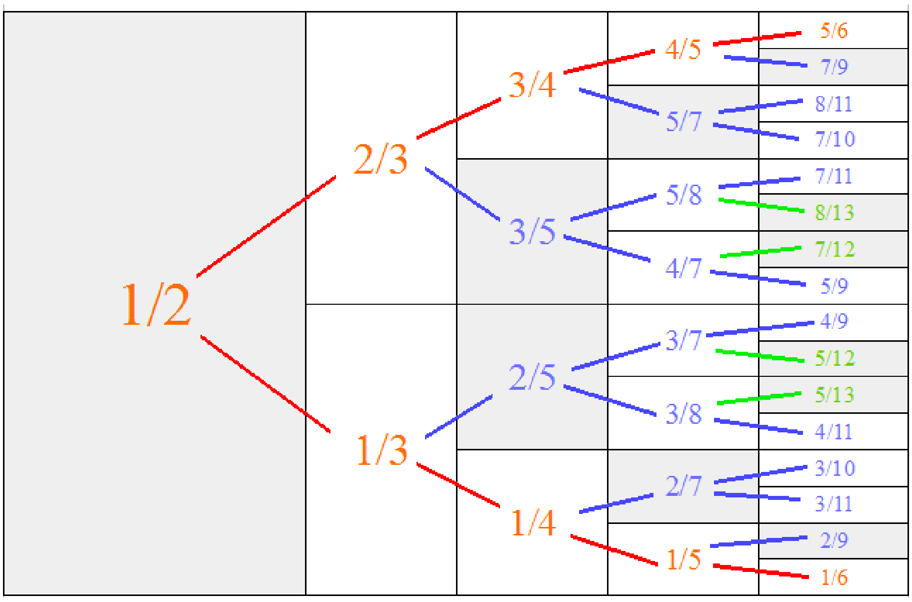


*Fig 2: The Stern-Brocot Tree*

The first order fractions are shown in red, the second order fractions in blue and the third order fractions in green.

Those fractions in the shaded boxes all have a NICF which ends in 1/2 and which can therefore be written in two different ways. They are the starting points for two branches of fractions all with the same order, one with positive remainders and the other with negative remainders. For example, starting at 2/5 you can either proceed towards 1/2 using positive remainders viz: 3/7 → 4/9 → 5/11 → … 1/2 or you can proceed in the opposite direction using negative remainders: 3/8 → 4/11 → 5/14 → … 1/3.

This diagram is known as the Stern-Brocot Tree and was discovered independently by Moritz Stern and Achille Brocot in the middle of the nineteenth century. Like the Mandelbrot map – with which it is closely related – it is full of surprises. Firstly, the tree contains every possible rational fraction in its lowest possible terms. This amazing fact actually follows directly from its construction in terms of nearest integer continued fractions because every rational number has a unique NICF and fractions with common factors have the same NICF as their reduced forms.

Secondly, you will see that the rule of Farey addition applies all over the tree. Follow any of the indicated branches and you will see the numerators and the denominators increasing step by step by the numerator and denominator of the fraction towards which the branch is heading. It is easy to see why. Suppose we have a NICF whose last two coefficients are $j$ and $k$ – in other words, a NICF which ends $(j \pm \frac{1}{k})...)))$ This can be evaluated as follows: $(\frac{jk \pm 1}{k})...)))$ .As you evaluate your way back to the beginning, there is never a place where $k$ has to be used again so the final expression will remain a linear function of $k$ divided by a linear function of $k$. In other words, any NICF whose final coefficient is $k$ will have the form $\frac{pk \pm r}{qk \pm s}$ where $k$ goes from 0 to $\infty$. The start of the sequence (when $k = 0$) is $r/s$ and the fraction towards which the sequence is heading (as $k \to \infty$) is $p/q$.

Now two adjacent members of the sequence have the form $\frac{pk \pm r}{qk \pm s}$ and $\frac{p(k+1) \pm r}{q(k+1) \pm s}$ . But the latter expression is the Farey sum of $\frac{pk \pm r}{qk \pm s}$ and $\frac{p}{q}$ . Q.E.D.

The connection between the Stern-Brocot tree and the rotation numbers is obvious enough but there is more.

The NICF of any first order fraction number $x$ less than 0.5 is represented as $0 + r$; numbers above 0.5 are represented in the form $1 - s$ ($r$ and $s$ are both < 0.5). The *complement* of these numbers (i.e. $1 - x$) will be represented as $1 - r$ and $0 + s$ respectively. The rule is simple. If the sign is positive you add 1 to the coefficient; if it is negative you subtract 1; then you reverse the sign.

This procedure can be carried out on the second order fractions as well. Consider the fraction 3/8. Its NICF is $0 + \frac{1}{3-\frac{1}{3}}$ . Its first order complement is $1 - \frac{1}{3-\frac{1}{3}}$ which is 5/8 but its second order complement is $0 + \frac{1}{2+\frac{1}{3}}$ which equals 3/7. 5/8 is the fraction which lies in the corresponding place in the upper half of the tree but 3/7 lies on the other side of the second order branch of which 2/5 is the head.

If we apply both first and second order complements at the same time we get $1 - \frac{1}{2+\frac{1}{3}}$ which equals 4/7.

Of course we can apply third order complements to third order fractions and fourth order complements to fourth order fractions etc. etc. In fact, every number between 0 and 1, even the irrational ones, has what I call its *total complement* in which every coefficient in its nearest integer continued fraction has been complemented. The result is what Douglas Hofstadter, in his amazing book Gödel, Escher, Bach, published in 1979, called the INT function (where INT stands for INTerchange, not integer). A graph of the INT function is shown in Fig 3. It is a fractal which has a finite discontinuity at every rational fraction. (Since irrational numbers have an infinitely long NICF the jumps there are zero)

The jumps at the first order fractions are indicated in red and the jumps at the second order fractions in blue. In the middle of every jump there is a dot which is the total complement of the fraction itself. The size of the jump may be taken as a measure of the *rationality* of the fraction. A histogram of the sizes of the first and second order jumps has been plotted at the bottom and in the background, a representation of the lobes around the cardioid of the Mandlebrot set, straightened out using a suitable transformation[2].

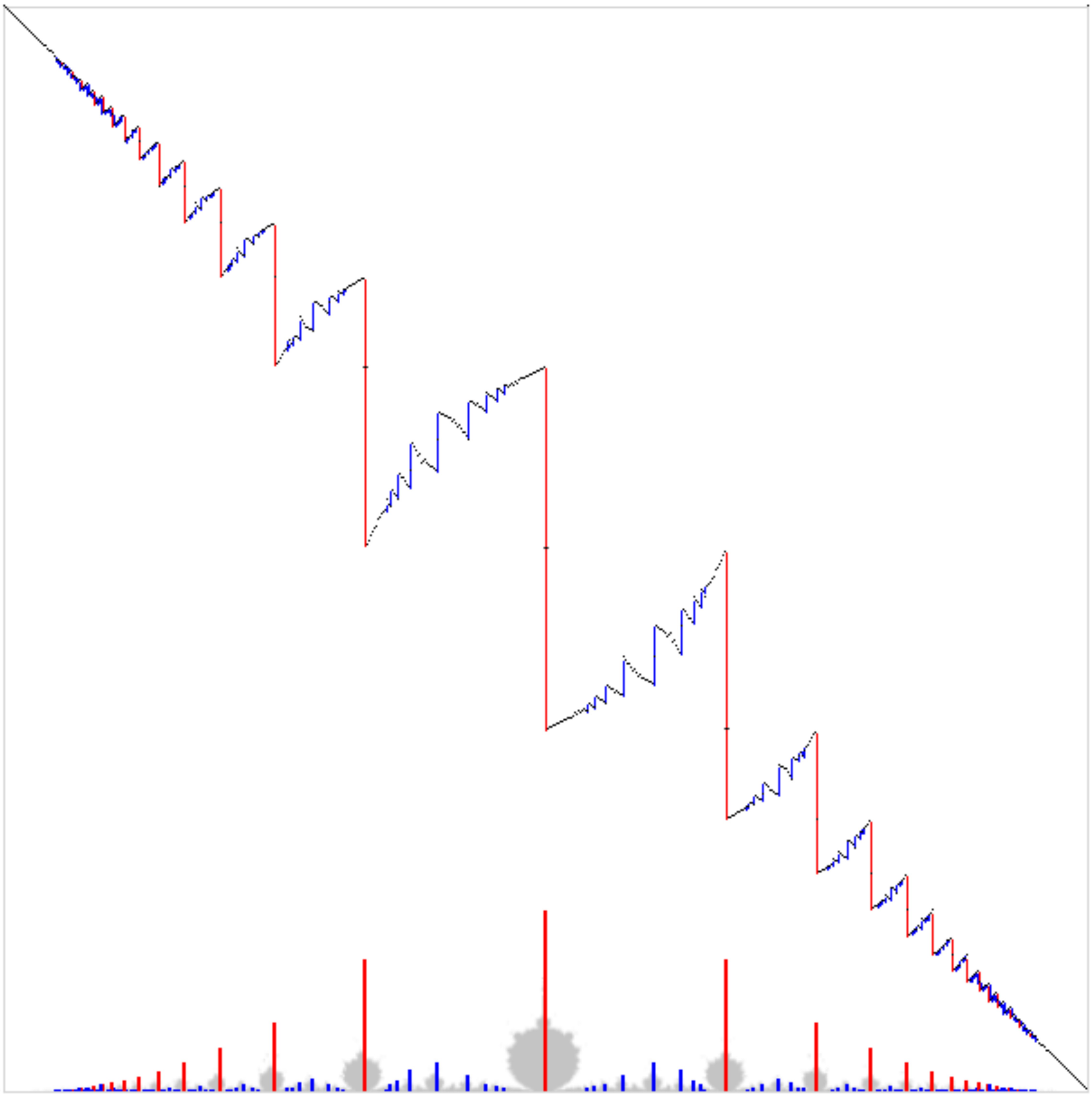

*Fig 3: Hofstadter's INT function showing the steps between the first and second order complements with the lobes of the Mandelbrot Set in the background*

We now have the clearest evidence that both the position and the size of the lobes of the Mandelbrot Set are intimately connected with the rationality of the numbers between 0 and 1.

To see why this is so we must remind ourselves how the Mandelbrot Set is generated. Formally it is defined as the set of all the points $c$ in the complex plane for which the point $z = 0$ remains within a finite bound under the iteration $z \Leftarrow z^2 + c$. More simply – choose a complex number $c$ ($= a + ib$); start at the origin; apply the transformation; (this first step will always take you to to the

2 The equation of the cardioid is $u = v - v^2$ where $u$ and $v$ are complex numbers. $v$ has a constant modulus of 0.5. A lobe at the point $u$ on the cardioid has a rotation number equal to the argument of $v$ (expressed as a fraction of a circle).

point *c*); repeat ad infinitum. If you find yourself homing in on a repeated pattern, colour the point *c* black. Better still, colour the point according to the periodicity of the pattern you find yourself in. Fig 4 shows the result.

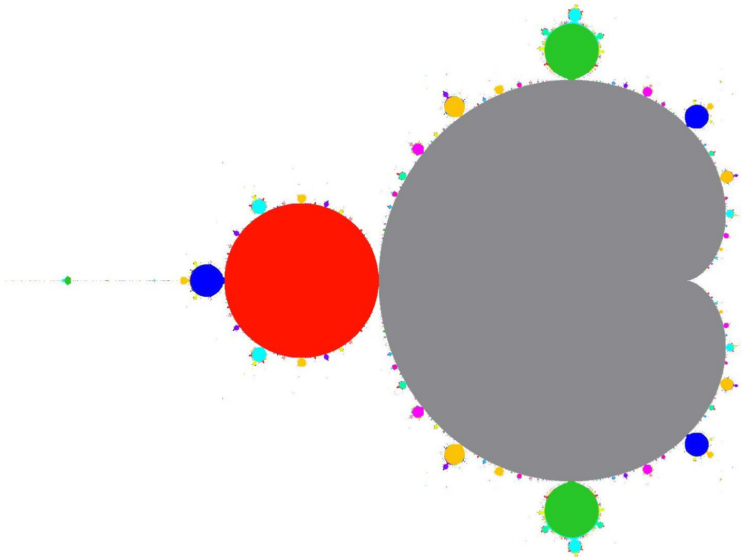

*Fig 4: The Mandelbrot Set showing the periodicity of the lobes.*
*Grey:1, Red:2, Green:3, Blue:4, Orange:5 ...*

The main cardioid has a periodicity of 1. What this means is that for all points *c* within this cardioid, *z* homes in on a single point somewhere in the plane. If *c* is inside lobe 2 (the red one) *z* eventually oscillates between two points. (If you move *c* along the negative X axis *z* will bifurcate at the junction between lobes 1 and 2.)

There are just two green lobes of period 3 attached to the main cardioid and just two of period 4 (blue) – but there are four lobes of period 5 shown in orange. If you look back at figure 1 you will see that the rotation numbers of the green lobes are 1/3 and 2/3, and that the rotation numbers of the blue lobes are 1/4 and 3/4. (There is, of course another lobe of periodicity 4 coloured blue, but it is not attached to the main cardioid. You could claim that its rotation number was 2/4 but this is, of course equal to 1/2.) The orange lobes have rotation numbers 1/5, 2/5, 3/5 and 4/5. There are no prizes for guessing the rule – the periodicity is the denominator of the rotation number!

So if the denominator is the periodicity, what is the significance of the numerator?

To answer this question, all we have to do is look at the first few iterations of the point *z* when *c* lies inside the two upper lobes of periodicity 5. Fig 5 shows exactly what is going on.

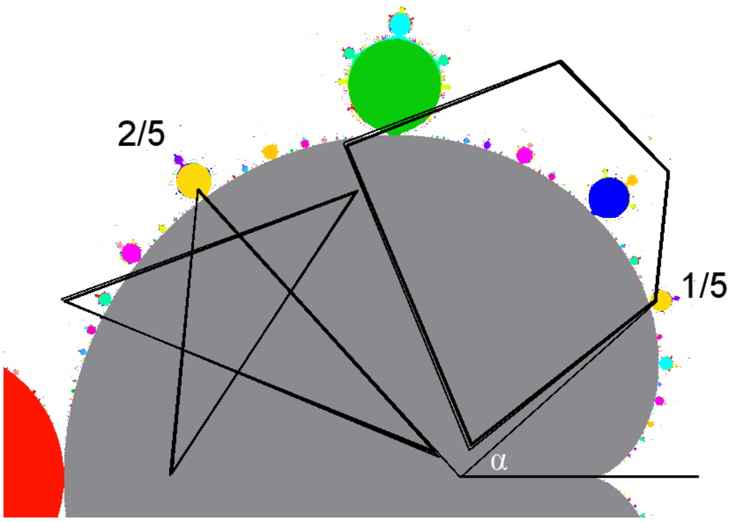


*Fig 5: The first few iterations of z when c = 1/5 and c = 2/5*

When $c$ equals a value inside the lobe whose rotation number is 1/5, $z$ jumps round an irregular pentagon; but when $c$ is inside lobe 2/5, $z$ jumps round a five pointed star i.e. jumping 2 steps at a time. As I have said, the first jump always goes to the point $c$. The angle α is the argument of $c$ and as you can see, it is basically the exterior angle of the pentagon which results. If $c$ lies inside the lobe whose rotation number is 2/5, the exterior angle is doubled and the figure traced out is a five pointed star. The numerator determines the size of the angle and hence the type of star traced out by $z$. I call this parameter the *step size* of the lobe. As $c$ moves round the edge of the cardioid, the pattern traced out by $z$ transforms through all the possible polygons and star shapes at every rational fraction. At the irrational numbers, The points reached by $z$ will lie on a closed curve but $z$ will never fall exactly on a point which it has already visited. (If you like, you could say that the periodicity of the 'lobe' is infinite and, of course, its diameter will be zero.)

In summary, we have shown that:

- *every lobe is associated with a rational fraction called its rotation number*
- *the denominator of the rotation number is the* period *of the lobe*
- *the numerator is its* step size
- *the position of the lobe on the cardioid is closely related to the* size *of the rotation number*
- *the size of the lobe is closely related to the* rationality *of the rotation number*
- *lobes fall into natural sequences of diminishing size whose rotation numbers correspond to sequences of nearest integer continued fractions in which the final coefficient ranges from 0 to ∞. These sequences are conveniently illustrated using the Stern-Brocot tree.*
- *adjacent lobes in each sequence obey the Farey Fraction rule*

One last point to mention is this. The Mandelbrot Set is generated using a specific non-linear equation $z \Leftarrow z^2 + c$ and it may be thought that all the above structure is a result of that particular equation. This is not so. The lobes of the Mandelbrot set are organised in the way that they are because that is the way the rational fractions are organised along the number line. If we use a different equation, our stable set may take on a very different shape – but wherever a parameter can sweep through all the rational fractions, the same sequence of structures will appear as shown for example in Fig 6.

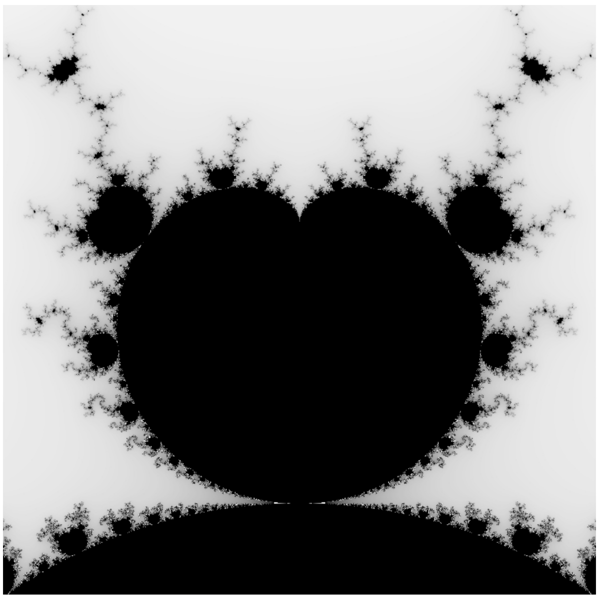

*Fig 6:* $z \Leftarrow z^3 + c$